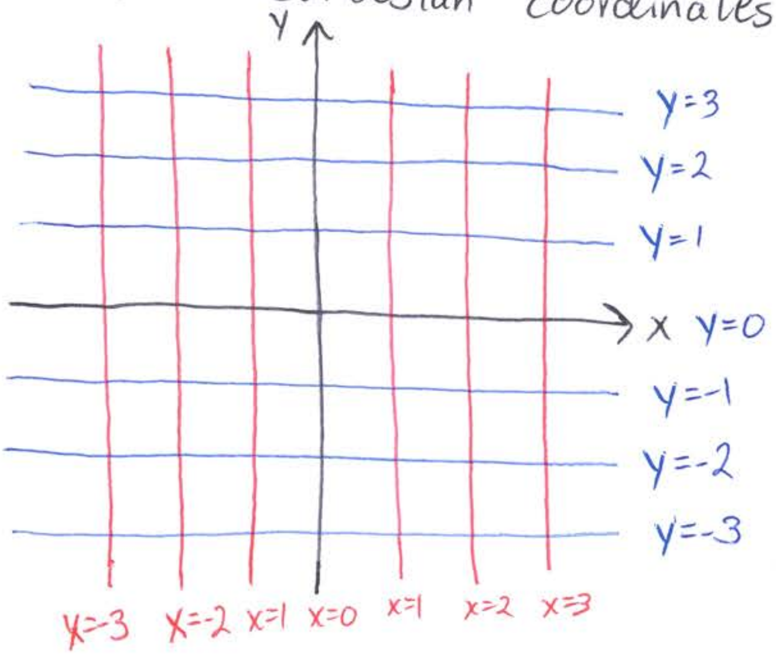
Lecture 34

10.3 - Polar Coordinates

Coordinates on the plane $\mathbb{R}^2$ are basically a choice of "grid lines" on the plane (grid lines are the graphs of equations "coordinate = constant"). For example, the grid lines for the usual Cartesian coordinates looks like:

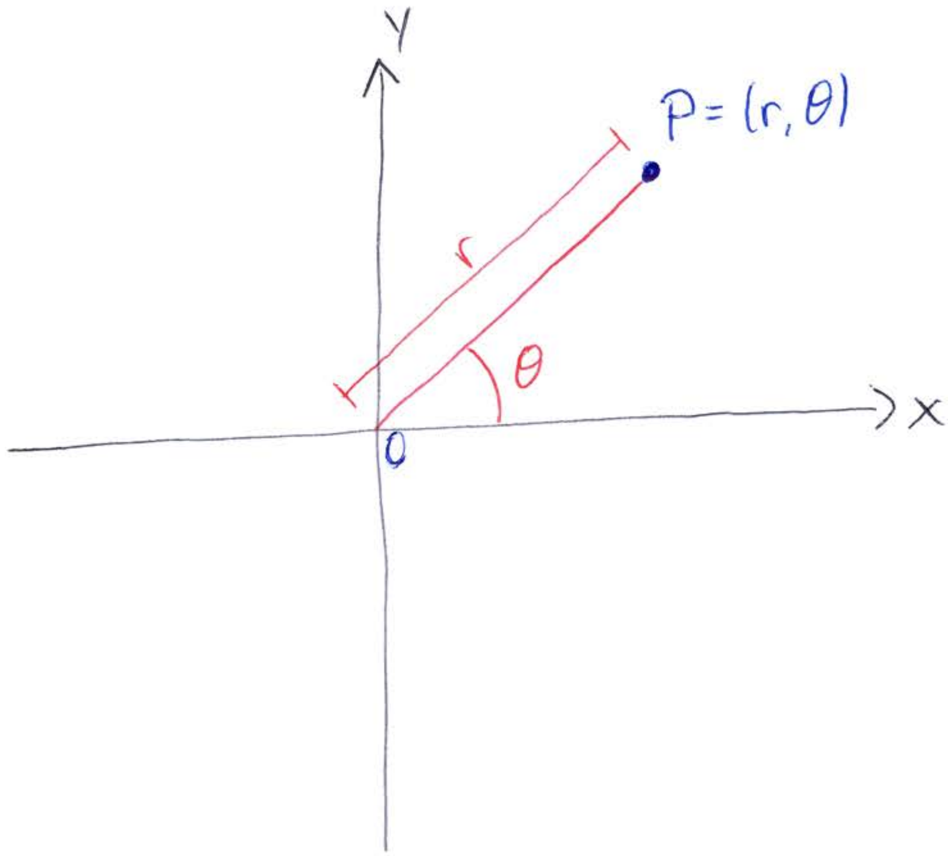


Polar Coordinates aims to describe the plane in terms of circles, centered at the origin. This gives 2 coordinate

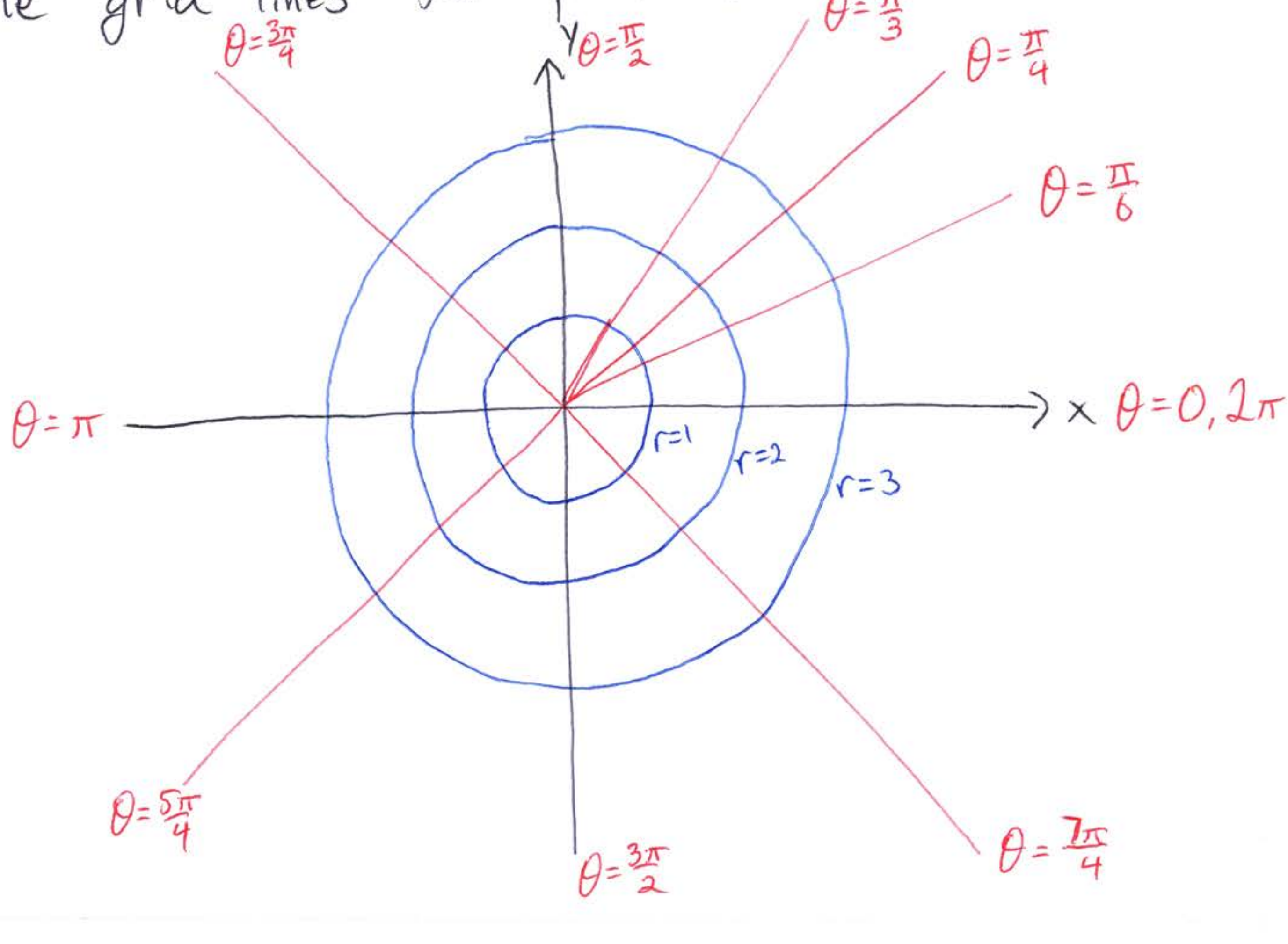
1) distance from the origin = r

2) angle of the ray connecting the point to the origin with the positive x-axis = θ

Graphically, these points look like:



The grid lines for polar coordinates look like:

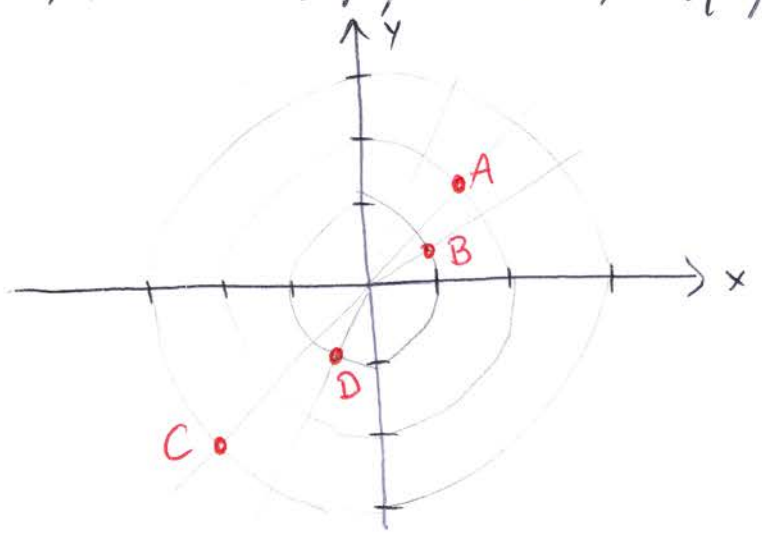


Notice that the graphs of:

- $r = c$ is a circle of radius c, centered at the origin
- $\theta = \alpha$ is a ray (half-line) starting at the origin and making an angle α with the positive x-axis.

Ex: Plot the points with polar coordinates:

$A = (2, \frac{\pi}{4})$, $B = (1, \frac{\pi}{6})$, $C = (3, -\frac{3\pi}{4})$



We can extend the definition of polar coordinates to include negative r values too:

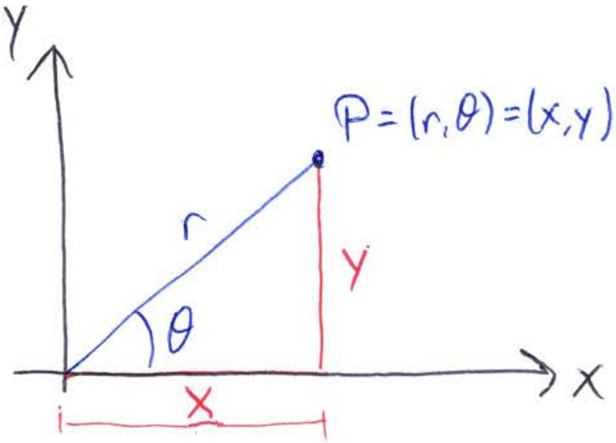
$$(-r, \theta) := (r, \theta + \pi)$$

So, if r is negative, the point (r, θ) is a distance $|r|$ from the origin, with angle $\theta + \pi$

Ex: Plot $D = (-1, \frac{\pi}{3})$ above.

Polar → Cartesian

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$



Ex

Convert $\left(3, -\frac{\pi}{3}\right)$ to Cartesian.

$$x = 3\cos\left(-\frac{\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$y = 3\sin\left(-\frac{\pi}{3}\right) = 3\left(-\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{2}$$

$$\left(3, -\frac{\pi}{3}\right) = (r, \theta) \to (x, y) = \left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$$

Cartesian → Polar

Using the above, we can find equations for r & θ in terms of x & y:

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$$\Rightarrow \boxed{r^2 = x^2 + y^2}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \boxed{\tan\theta = \frac{y}{x}}$$

Ex: Convert ⓐ $(1, \sqrt{3})$ and ⓑ $(-\sqrt{2}, \sqrt{2})$ to polar coordinates.

ⓐ $r^2 = (1)^2 + (\sqrt{3})^2 = 4$

$\Rightarrow r = 2$

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\Rightarrow \theta = \pi/3$ or $4\pi/3$

Point is in first quadrant $\Rightarrow \theta = \frac{\pi}{3}$

$(1, \sqrt{3}) \rightarrow (r, \theta) = \left(2, \frac{\pi}{3}\right)$

ⓑ $r^2 = (-\sqrt{2})^2 + (\sqrt{2})^2 = 2 + 2 = 4 \Rightarrow r = 2$

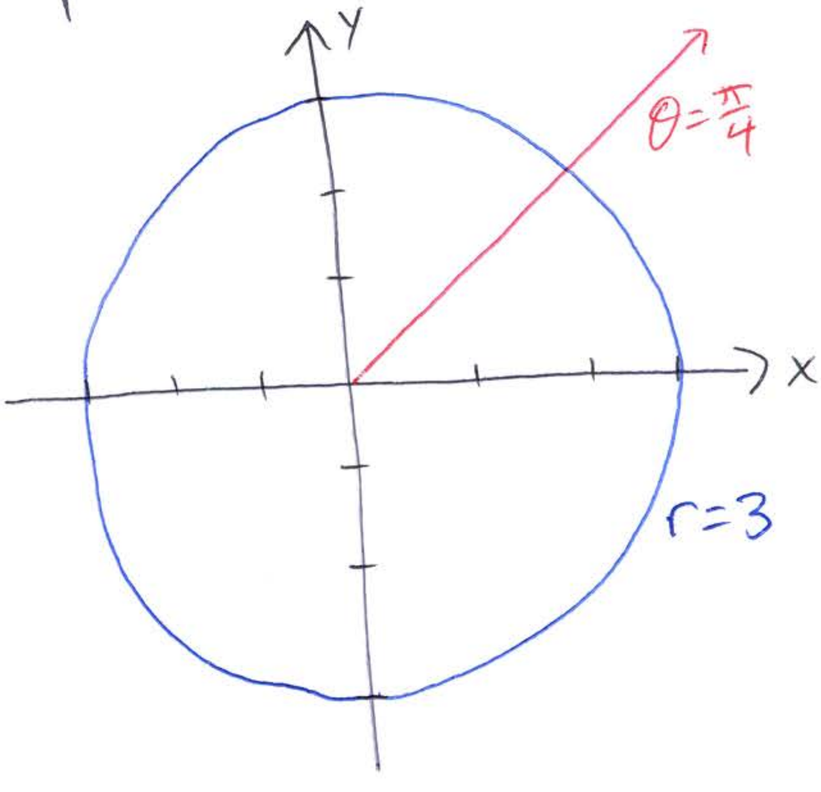
$\tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1 \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

Point is in second quadrant $\Rightarrow \theta = \frac{3\pi}{4}$

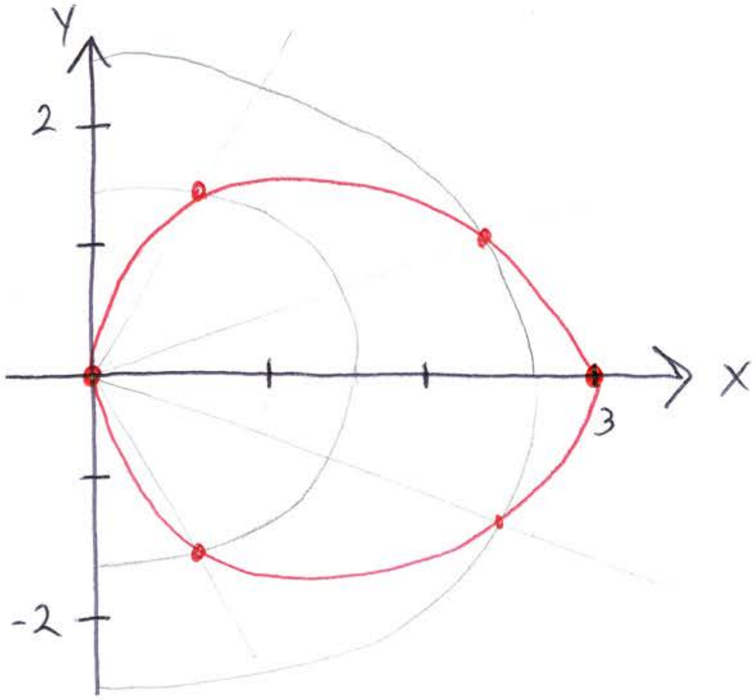
$(-\sqrt{2}, \sqrt{2}) \rightarrow (r, \theta) = \left(2, \frac{3\pi}{4}\right)$

Graphing In Polar Coordinates

Ex: Graph $r = 3$ & $\theta = \pi/4$



Ex: Graph the equation $r = 3\cos\theta$.
What is an equation for this in Cartesian coordinates?



θ	r
0	3
$\pi/3$	$3/2$
$\pi/2$	0
$2\pi/3$	$-3/2$
π	-3
$\pi/6$	$3\sqrt{3}/2$
$5\pi/6$	$-3\sqrt{3}/2$

(Scaling on my axes is a bit bad, but this is a circle.)

$$r = 3\cos\theta \iff \underline{r^2} = 3\underline{r\cos\theta}$$

$$\Rightarrow x^2 + y^2 = 3x$$

$$\Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = 0 + \frac{9}{4}$$

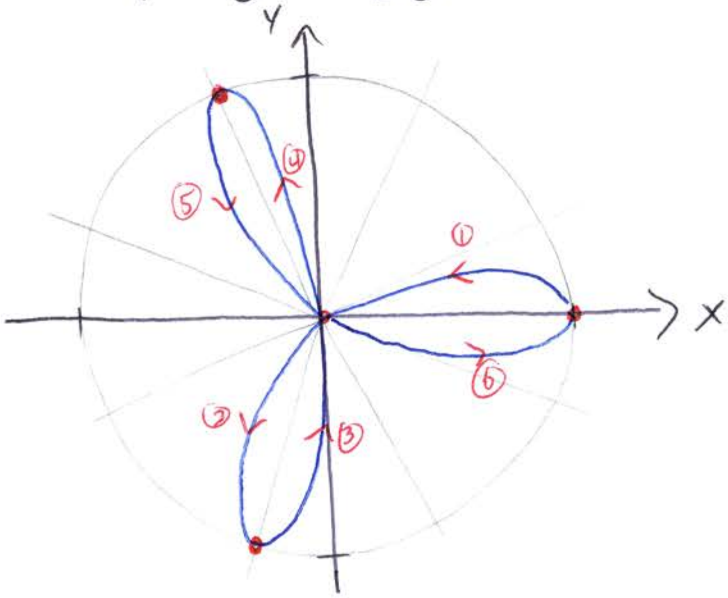
$$\Rightarrow \boxed{\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}}$$

Polar equations can create some very interesting graphs:

Ex: Sketch the curve

$$r = \cos 3\theta$$

θ	r
0	1
$\pi/6$	0
$\pi/3$	-1
$\pi/2$	0
$2\pi/3$	1
$5\pi/6$	0
π	-1

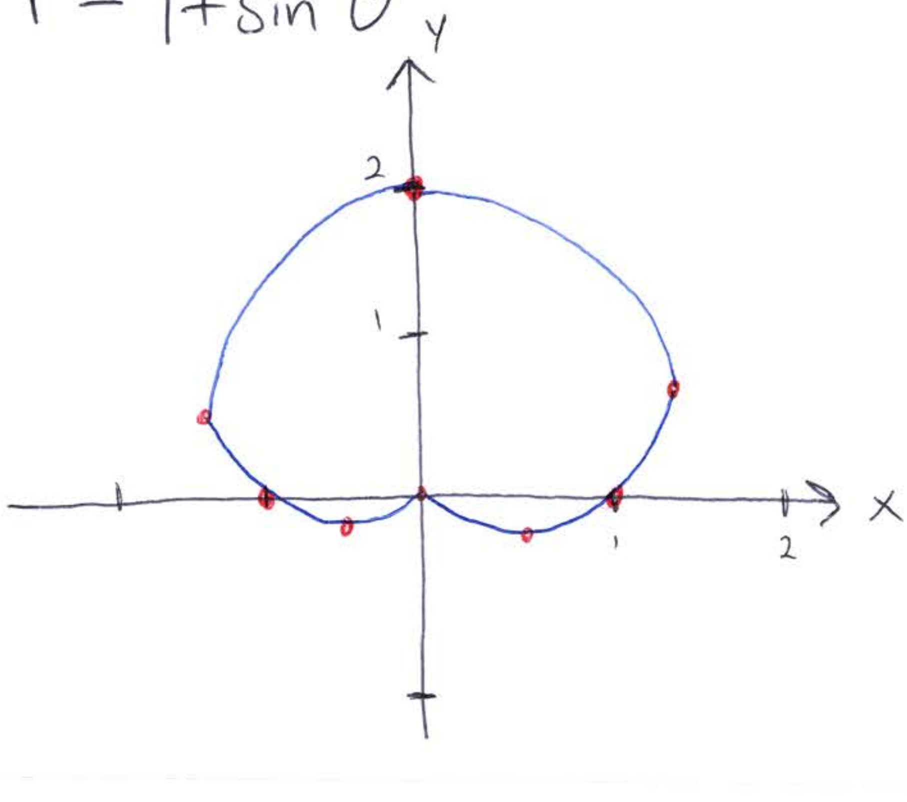


#'s & arrows indicate how curve moves.

Ex: Sketch the curve

$$r = 1 + \sin\theta$$

θ	r
0	1
$\pi/6$	$3/2$
$\pi/2$	2
$5\pi/6$	$3/2$
π	1
$7\pi/6$	$1/2$
$3\pi/2$	0
$11\pi/6$	$1/2$
2π	1



Tangents to Polar Curves

If we have a curve $r=f(\theta)$ in polar coordinates, we can create parametric equations out of it: $x = r\cos\theta = f(\theta)\cos\theta$

$$y = r\sin\theta = f(\theta)\sin\theta$$

So, we can find

$$\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$

<u>Ex</u>: Consider the curve $r = 3\cos\theta$.

ⓐ Where are vertical tangents?

$\frac{dy}{d\theta} = \frac{d}{d\theta}(3\cos\theta\sin\theta) = 3(-\sin^2\theta + \cos^2\theta) = 0$ when $\theta = \frac{\pi}{4} + \frac{n\pi}{2}$

$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos^2\theta) = -6\cos\theta\sin\theta = 0$ when $\theta = \frac{n\pi}{2}$

Vertical tangents @ $\theta = 0, \frac{\pi}{2}$ (others are repeats)

ⓑ Where are horizontal tangents?

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ (others are repeats)

ⓒ Find the tangent line at $\theta = \pi/3$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{3(-3/4 + 1/4)}{-6(\sqrt{3}/4)} = \frac{-3/2}{-3\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$\theta = \pi/3 \to r = 3/2 \to (x,y) = (3/4, 3\sqrt{3}/4)$

$$\boxed{y - \frac{3\sqrt{3}}{4} = \frac{1}{\sqrt{3}}\left(x - \frac{3}{4}\right)}$$